A GENERAL THEORY OF THE GLOW-LAMP. II.[1]

By H. S. Weber.

III.

THE definitive quantity in the case of the glow-lamp is the energy which the electric current develops in the filament in a unit of time. By means of this the temperature of the filament is determined when the surface is known, and thus finally the quantity of light which the filament radiates is given. One of the fundamental questions in the domain of glow-lamp illumination is that of the relation between the amount of light radiated and the expenditure of energy necessary to its production. Voit has attempted to find a simple expression for this quantity based upon experiments with incandescent lamps made at the Munich electrical exhibition. As has already been pointed out he found that the light radiated was neither proportional to the square nor to the fourth power, but approximately proportional to the cube of the energy consumed. "The formula $H=qE^3$ may be regarded as an empirical rule which for glow-lamps is between about 6 c.p. and 120 c.p." For the constant q, Voit found, from the Munich measurements: —

Edison lamp	(16 c.p.) q =	37.6×10^{-6}	Siemens lamp	(16 c.p.) q =	22.5×10^{-6}
Edison lamp	(8 c.p.) =	110.6×10^{-6}	Small Müller lamp	(20 c.p.) =	21.3×10^{-6}
Small Swan lamp	(10 c.p.) =	84.8×10^{-6}	Medium Müller lamp	(50 c.p.) =	6.7×10^{-6}
Large Swan lamp	(40 c.p.) =	9.6	Large Müller lamp	(100 c.p.) =	2.1×10^{-6}
Maxim lamp	(28 c.p.) =	14.8	Cruto lamp	(10 c.p.) =	25.0×10^{-6}

The constant q according to these measurements is very different in the case of different lamps, and as Voit has pointed out, it is small in proportion as the useful light energy of the lamp increases. No further investigation concerning the factors which

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are involved in this coefficient was attempted by Voit, and nothing has since been done in this direction. In my measurements of thirty-three types of glow-lamps, I have found for every lamp Voit's empirical relation, according to which the intensity of light is proportional to the cube of the energy consumed, fairly well established. It appeared, however, that the size of Voit's coefficient is not quite a constant, but that it increases as the light increases from small intensities, then more rapidly with increasing brightness to a maximum, and finally diminishes again.

A glance at the column marked $q=\frac{H}{E^3}$ of the last four tables shows the change in this quantity with increasing HE as just described. At the same time it appeared from measurements, as Voit had shown, that the size of the coefficient for different lamps is enormously different. As a mean value of q we find from the above tables:—

For the Woodhouse lamp	41.0×10^{-6} and 32.0×10^{-6}
For the Sunbeam lamp	2.12×10^{-6} and 1.49×10^{-6}
For the lamp of the Allgemeinen Electricitäts-Gesellschaft	87.9×10^{-6} and 70.4×10^{-6}
For the Cruto lamp	75.7×10^{-6} and 55.8×10^{-6}

where H is the mean horizontal intensity or the mean spherical intensity respectively. It is easily seen that all these phases are represented in a formula for radiation already presented. The intensity of any given visible homogeneous ray of wave-length λ is given through the formula:—

$$s=I\frac{c}{\lambda^2}\pi F e^{aT-\frac{1}{b^2\lambda^2 T^2}}.$$

The brightness of this radiation is proportional to that quantity. In order to obtain an expression for the total brightness, we may take advantage of the fact that the candle-power of a solid glowing body increases in proportion to the brightness of the homogeneous ray, of which the wave-length is about 0.54μ. If we set in this last equation in place of λ this value, then the expression gives us the relation between the total brightness and the temperature.

The energy which is consumed in the glowing filament when the same has been warmed to a stationary temperature T is, according to the above equation,

$$E=ICFTe^{aT}.$$

If we take cognizance of the fact that the constant of total radiation has a value of

$$\tfrac{1}{2}\pi\sqrt{\pi}bc,$$

we obtain as an expression for the quotient H/E^3 the following formula: —

$$\frac{H}{E^3}=\frac{m}{C^2F^2}\cdot\frac{1}{T^3\cdot e^{2aT+\frac{1}{b^2\lambda^2T^2}}}=\frac{m}{C^2F^2}\phi(T), \tag{6}$$

in which small m is a constant. In order to determine the function $\phi(T)$, we must know, aside from the quantities $a=0.0043$ and $\lambda=0.54$, the constant b^2 for carbon. From Langley's observations concerning the distribution of energy in the spectrum of black carbon at a temperature of 178°, I find the quantity $b^2=0.192\times10^{-6}$, where the wave-lengths were measured in microns. From my own measurements upon the change in brightness of homogeneous rays from the gray carbons of the Sunbeam lamp, I determined for the temperature interval $T=1450°\cdots T=1650°$, $b^2=0.190\times10^{-6}$. If we assume for b^2 the latter value, then the quantity $b^2\lambda^2$ is equal to 0.0052×10^{-6}.

In general, the function depends in a complicated way upon the temperature. If, however, we have to consider only the temperature interval from 1400° to 1600°, within which range the temperature of the glow-lamp lies, it may be shown that the character of this function for the interval in question is very simple. The following table shows the change of $\phi(T)$ for successive intervals of 10°: —

T	lg $\Phi(T)$	$\Phi(T)$	T	lg $\Phi(T)$	$\Phi(T)$
1440°	−42,959	0.953 × A	1520°	−42,913	0.998 × A
1450	−42,945	0.966	1530	−42,916	0.995
1460	−42,934	0.977	1540	−42,920	0.991
1470	−42,924	0.987	1550	−42,927	0.984
1480	−42,918	0.993	1560	−42,936	0.975
1490	−42,914	0.997	1570	−42,947	0.965
1500	−42,912	0.999	1580	−42,959	0.953
1510	−42,911	1.000			

The last vertical column of this table indicates that the function $\phi(T)$ between $T=1400°$ and $T=1580°$ goes through precisely the same changes as the factor q already described. It increases slowly from 1400° on, reaches a maximum of 1510°, the value of which is expressed by the equation

$$3+2aT-\frac{2}{b^2\lambda^2T^2}=0,$$

and then diminishes slowly. Between 1475° and 1542° the value of $\phi(T)$ changes only 1 per cent; between 1463° and 1555°, 2 per cent; from 1554° to 1565°, 3 per cent.

Observations with the various individual lamps are in entire accord with this result; viz. the maximum for q in the case of the Woodhouse lamp occurs in the neighborhood of 1415°; for the Sunbeam lamp, in the neighborhood of 1498°; for the Berlin lamp, 1503°; and for the new Cruto lamp, finally, at 1514°.

We may safely assume, then, that if the value of the function $\phi(T)$ for a temperature of 1510° is A, the quantity $q=HE^3$, for the entire range of temperature between 1452° and 1465°, will change at most 3 per cent from the constant value mAC^2F^2. Further, since the maximum value of q is

$$q_{max}=\frac{mA}{C^2F^2},$$

the value of $q_{max}C^2F^2$ must be constant for all lamps. This relationship is found to be true in fact, as the following tables for the four lamps will show:—

	q_{max}	F^2	C^2	$q_{max}C^2F^2$
Woodhouse lamp .	33.10×10^{-6}	0.790 cm.^{4}	0.285×10^{-9}	7.3×10^{-15}
Sunbeam lamp . .	1.57	25.400	0.172	6.9
Berlin lamp . . .	72.30	0.577	0.167	7.3
New Cruto lamp. .	56.50	0.399	0.292	6.6

The small variation for the value given in the last vertical column of this table is easily accounted for, since these quantities involve the square surface of the filament, which is difficult to determine, and the square of the constant of total radiation.

The mean value of this quantity ($q_{max}C^2F^2$) I found for all the carbons investigated to be 6.9×10^{-15}.

Since q changes only about 1 per cent between 1460° and 1560°, we may assume with a high degree of approximation for this interval, for all lamps, the relation

$$H = \frac{6.8 \times 10^{-15}}{C^2F^2} \cdot E^3;$$

where H is the mean spherical candle-power expressed in British candles.

IV.

In its application to the glow-lamp, the relationship between the energy consumed in the filament in a unit of time, and the resulting brightness of radiation, is the most important quantity. Ever since the first measurements upon lamps were made it has been known that with increasing temperature of the filament this quantity diminishes very rapidly. But the law of the relationship between this quantity and the temperature has not been accurately known. The formulas which we have already developed from our observations enable us to express this relation in a simple manner.

Since the energy expended in a unit of time is

$$E = ICFTe^{aT}, \text{ where } C = \tfrac{1}{2}\pi\sqrt{\pi}\, cb,$$

and since the brightness expressed in any units — for example, British candles — is given by the expression

$$H = \alpha\frac{c}{\lambda^2} \cdot \pi F e^{aT - \frac{1}{b^2\lambda^2T^2}},$$

in which α is a factor of proportionality, and λ is the value 0.54, it follows that the quantity $E_1 = \dfrac{E}{H}$ is given by the equation

$$E_1 = \frac{I\pi\lambda\sqrt{\pi}}{2\alpha} \cdot b\lambda T . e^{\frac{1}{b^2\lambda^2T^2}}. \tag{7}$$

The energy necessary to develop a unit of light in a unit of time is consequently proportional to the temperature of the function, and depends upon the value of

$$\Psi(b\lambda T) = b\lambda T e^{\frac{1}{b^2\lambda^2T^2}}.$$

The change of this temperature of the function for the given value of $b^2\lambda^2 = 0.0552 \times 10^{-6}$ for the interval of temperature applicable to the glow-lamp, 1400° to 1650°, is given in the following tables for intervals of ten degrees. At the same time values of this function for the higher temperatures, 1700°, 1800°, 1900°, and 2000°, are given, in order to indicate the further progress of this function at higher temperatures.

T	$\Psi(b\lambda T)$	T	$\Psi(b\lambda T)$	T	$\Psi(b\lambda T)$
1400	3501	1500	1135	1600	455
1410	3093	1510	1027	1610	419
1420	2740	1520	931	1620	387
1430	2433	1530	846	1630	358
1440	2166	1540	770	1640	332
1450	1934	1550	702	1650	308
1460	1731	1560	642	1700	214
1470	1552	1570	588	1800	115
1480	1395	1580	539	1900	68
1490	1257	1590	495	2000	44

On account of the great importance of the quantity E, an experimental proof that E is in fact such a function of the temperature, as it is represented to be by Equation 7, is of considerable value. For the purposes of this proof we may make use of the contents of the tables which have already been given to illustrate the properties of the four different glow-lamps. If we seek by means of these tables the values at the temperatures indicated, and divide these values by the value of $\Psi(b\lambda T)$, for the same temperatures we should obtain a constant quotient in case E really is of the form given in Equation 7.

The numbers given in the last columns of the following four tables are these quotients. It will be seen that for each of the four lamps the fluctuations are such as to size and direction that they may well be ascribed to errors of observation.

LAMP OF THE BERLIN CO. (ALLGEMEINE ELEK.-GESELLSCHAFT).

E_1	T	$\Psi(b\lambda T)$	$\frac{\Psi(b\lambda T)}{E_1}$
10.87 W.	1464°	1659	153
8.67	1483	1354	156
7.07	1503	1103	156
5.88	1522	914	155
4.91	1541	763	155
4.18	1557	640	158
3.57	1574	568	158
3.09	1591	491	158
2.71	1607	430	158
2.41	1621	384	159

SUNBEAM LAMP.

E_1	T	$\Psi(b\lambda T)$	$\frac{\Psi(b\lambda T)}{E_1}$
9.64 W.	1463°	1677	174
8.26	1478	1426	173
6.70	1498	1159	173
5.20	1524	897	173
4.11	1549	709	172
3.27	1573	573	176
2.68	1595	475	177
2.25	1615	403	179
1.94	1636	342	176
1.69	1654	299	177

NEW CRUTO LAMP.

E_1	T	$\Psi(b\lambda T)$	$\frac{\Psi(b\lambda T)}{E_1}$
14.03 W.	1434°	2326	166
10.63	1460	1731	163
8.41	1482	1367	163
6.08	1514	989	163
5.09	1532	831	163
4.32	1550	702	163
3.70	1566	610	165
3.14	1582	530	169
2.66	1600	455	171
2.31	1618	393	170

LAMP OF WOODHOUSE AND RAWSON.

E_1	T	$\Psi(b\lambda T)$	$\frac{\Psi(b\lambda T)}{E_1}$
20.60 W.	1400°	3501	175
12.37	1441	2141	173
8.20	1479	1411	172
5.77	1514	989	172
4.21	1548	716	170
3.63	1561	637	175
3.17	1578	549	173
2.76	1592	487	176
2.55	1606	433	170
2.23	1620	387	173

The contents of these four tables can also be used to demonstrate that the four different filaments to which they apply have the same or nearly the same values for the constant b. If b^2 is the same for these four carbons and equal to 19×10^{-6}, then the quotients must be the same for all four lamps. In order to compare these quotients among themselves, however, it is first necessary to take cognizance of the question of spherical intensity, since these lamps have very different relations between the mean spherical and the mean horizontal candle-power, as is shown by the following tables: —

	$\Psi(b\lambda T) : E_1$	R	$R\Psi(b\lambda T) : E_1$
Berlin lamp	157	0.801	126
Sunbeam lamp	175	0.711	124
New Cruto lamp	165	0.734	121
Woodhouse lamp	173	0.780	135

If we apply to each lamp its proper reduction factor, we find that the correct value of the quotient given in the last column is nearly the same for the four types. It follows from the equation for E_1, already given, that

$$-\frac{dE_1}{E_1} = \frac{db}{b} \cdot \left(\frac{2}{b^2\lambda^2 T^2} - 1\right),$$

a result which shows that the percentage of decrease in E_1, which is caused by a given percentage of diminution in b, is as much larger than the factor

$$\left(\frac{2}{b^2\lambda^2T^2}-1\right)$$

as the latter is larger than 1. This factor, however, has the following values (when $b^2\lambda^2=0.0552\times 10^{-6}$); viz.

17.48 when $T=1400^\circ$	14.08 when $T=1550^\circ$
16.23 when $T=1450^\circ$	13.15 when $T=1600^\circ$
15.10 when $T=1500^\circ$	12.31 when $T=1650^\circ$

The change in the constant b of n per cent of its value would also produce a change in the value of E_1, which change would be from 12 to 17 times as great.

The tables given can also be made to furnish an answer to the question of the percentage of decrease in E_1 when the temperature rises one degree.

From Equation 7 it follows that

$$-\frac{dE_1}{E_1}=\frac{dT}{T}\left(\frac{2}{b^2\lambda^2T^2}\right)-1,$$

from which we can calculate the diminution of E_1 for one degree rise of temperature. The result is as follows:—

$\frac{1}{14}\times 17.48\% = 1.24\%$ for 1400°	$\frac{1}{15.5}\times 14.08\% = 0.91\%$ for 1550°
$\frac{1}{14.5}\times 16.23 = 1.12$ for 1450°	$\frac{1}{16}\times 13.15 = 0.82\%$ for 1600°
$\frac{1}{15}\times 15.10 = 1.01$ for 1500°	$\frac{1}{16.5}\times 12.31 = 0.75\%$ for 16.50°

Within the interval of temperature of 1400° to 1650° a rise in temperature of one degree produces, very nearly, a diminution of one per cent in the amount of energy necessary for the production of a unit of light. For this interval of temperature, therefore, a rise of n degrees in temperature will improve the economy of light production by nearly n per cent.

V.

The result of the discussion in Section III., according to which the total brightness of the glow-lamp may be expressed by means of the constant, $\alpha=6.8\times 10^{-16}$, the square of the radiating surface, the square of the constant C, the total radiation, and the cube of the expenditure of energy, gives us the means of determining the dimensions which a carbon filament must have in order to give a certain intensity of light under prescribed conditions. In order to calculate the length and the radius of the filament, which we will assume is of circular cross-section, we must know the difference of potential at which the lamp is to be used, the candle-power H which is to be produced, the temperature at which this candle-power is to be delivered, and finally the nature of the carbon, with required specific resistance, and the constant C of total radiation. Since the consumption of energy per unit of total brightness is determined by the temperature, we can set in place of the temperature a given value for E_1. From these six quantities α, ΔP, H, E_1, w, and C, the dimensions of l and ρ may be determined in the following manner:—

$$H=\frac{\alpha}{C^2F^2}\cdot E^3\text{; or since } E=E_1\cdot H,$$

$$H=\frac{\alpha}{C^2F^2}E_1^3H^3 \text{ and } F_2=\frac{\alpha}{C^2}E_1^3H^2,$$

or

$$l^2 4\pi^2\rho^2=\frac{\alpha}{C^2}\cdot E_1^3H^2.$$

The resistance of the carbon is

$$\frac{lw}{\pi\rho^2}=\frac{\Delta P^2}{E}=\frac{\Delta P^2}{E_1H}.$$

From these two equations it follows that

$$l=\sqrt[3]{\frac{\alpha H\cdot E_1^2\cdot\Delta P^2}{4\pi w C^2}};$$

$$\rho=\sqrt[6]{\frac{\alpha w^2\cdot H^4\cdot E_1^5}{4\pi^4\cdot C^2\cdot\Delta P^4}}$$

If the cross-section of the filament be rectangular, with one side a and the other b, and if $a=bn$, where b is a given number, the equations for the determinations of b and a are as follows: —

$$l=\sqrt[3]{\frac{\alpha H E_1^3 \cdot \Delta P^2}{\frac{4}{n}(1+n)^2 w C^2}};$$

$$a=\sqrt[6]{\frac{\alpha w^2 \cdot H^4 E_1^5}{4(n+n^2)^2 \cdot C^2 \Delta P^4}}.$$

In making use of these operations it is to be noted that the assumed value of the constant α, 6.8×10^{-15}, applies to mean spherical candle-power, and that the specific resistance of the carbon is to be taken as the resistance per cubic centimeter, the dimensions of the filament being expressed in centimeters.

As an example of the usefulness of this formula the following illustration may serve: A Siemens lamp, the carbon of which had a specific resistance $w=0.00209$ at its normal temperature, and a radiation coefficient $C=0.0000129$, gave 16 mean horizontal candles when the potential difference was 98 volts. The efficiency was 3.4 watts per candle. The reduction factor of this lamp to mean spherical candle-power was 0.71. In this case, then, H was equal to 11.36, and E_1 to 4.79. The measured length of the filament was found to be 15.58 centimeters, and the radius, measured at nine points equally distant along the thread, was 0.00756 cm. According to the formulas just given for l and ρ, l should be 15.74 cm., and ρ should be 0.0077 cm.

The measured dimensions were somewhat smaller than the calculated, which may result from the fact that the measurements were made at ordinary temperatures.

VI.

The general expression for radiation affords an example of the formula for the optical efficiency of the glow-lamp (the relation between the visible energy of radiation and the total radiation). The total energy of the radiation is

$$S'=\int_{\lambda_1}^{\lambda_2} \frac{c}{\lambda^2} \cdot \pi F \cdot e^{aT-\frac{1}{b^2\lambda^2 T^2}} \cdot d\lambda,$$

where λ_1 and λ_2 are the wave-lengths of the borders of the visible spectrum. A further development of this form gives:—

$$S' = c\pi FbTe^{aT}\left\{\int_{\frac{1}{b\lambda_2 T}}^{\infty} e^{-x^2}\cdot dx - \int_{\frac{1}{b\lambda_1 T}}^{\infty} e^{-x^2}dx\right\}.$$

If we take as wave-length boundaries of the visible spectrum the values 0.38 μ and 0.78 μ; and if we set $b^2 = 0.19\times 10^{-6}$, and assume as our interval of temperature 1400° to 1650°, the lower limit of the second integral will fall between 4.07 and 3.46, and as a consequence the value of the second integral for all these values will be smaller than 10^{-7}. S' may then be reduced to the following form:—

$$S' = c\pi FbTe^{aT}\int_{\frac{1}{b\lambda_2 T}}^{\infty} e^{-x^2}dx.$$

The energy of the total radiation, according to Equation 3, is

$$S = c\pi FbT\frac{\sqrt{\pi}}{2}\cdot e^{aT},$$

so that the expression for the optical efficiency is

$$N = \frac{S_1}{S} = \frac{2}{\sqrt{\pi}}\cdot\int_{\frac{1}{b\lambda_2 T}}^{\infty} e^{-x^2}dx.$$

For $T = 1500°$ the value of this integral is 0.0049, and $N = 0.55\%$.
For $T = 1550°$ the value of this integral is 0.0065, and $N = 0.73\%$.
For $T = 1600°$ the value of this integral is 0.0083, and $N = 0.94\%$.
For $T = 1650°$ the value of this integral is 0.0104, and $N = 1.17\%$.

For normal temperatures of glow-lamps, therefore, the efficiency is in the neighborhood of 1 per cent.

An experimental determination of this kind gives from 4 per cent to 5 per cent. (Measurements made by Blattner under my direction in 1885 gave about 5 per cent. Those made later by Merritt gave about 4 per cent.) These, however, were dependent upon the assumption that an aqueous solution of alum absorbed every part of the dark heat and let through all the visible rays. Since, however, this is not quite true, and these liquids let through some of the dark rays which lie near the boundary of the visible spectrum, the values experimentally obtained must be too large. If these solutions, for example, absorbed only those dark rays the wave-lengths of which are greater than (1.0 μ), the calculated value for the efficiency would be

$$N = \frac{2}{\sqrt{\pi}} \cdot \int_{\frac{1}{bT}}^{\infty} e^{-x^2} \cdot dx,$$

and the numerical values of the efficiency as follows:—

$T = 1500°$; $N = 3.0\%$.
$T = 1550°$; $N = 3.7\%$.
$T = 1600°$; $N = 4.3\%$.
$T = 1650°$; $N = 4.9\%$.

VII.

It has long been known that the rapid diminution in the life of glow-lamps with rising economy is to be explained in the fact that carbon at a certain temperature begins to suffer notable vaporization, which with further rise of temperature increases in intensity. After having found it possible to determine the temperature of carbon filaments for every condition of incandescence, it seemed to me important to fix the temperature at which carbon begins to show evaporation, and to show whether this temperature is the same for all sorts of filaments. The investigations which I have made in this direction are based upon the observations of the resistance of filaments. The resistance of the carbon filament remains constant at a given temperature as long as its cross-section is maintained. When the temperature is reached, however, at which appreciable quantities of carbon evaporate from the surface of the filament, then the resistance suffers a slow increase with time. If we raise the temperature of the filament, maintaining it constant at each step for a considerable time, and notice during this time the change of resistance, we may obtain an estimate of the temperature at which a considerable increase of resistance occurs.

As time intervals, during which the chosen temperatures were to be held constant, I selected thirty hours. By means of accumulators of large capacity, it is not at all difficult to regulate the strength of the current, so that the energy developed in the filament will remain constant to within one-thousandth of its value. From time to time during the test the current strength and the difference of potential were read, and the resistance was calculated from these values. The observations were thus continued until a temperature was found at which a sensible steady increase of resistance with the time became apparent. As an example of these measurements the following list of observations may be taken. They refer to the Woodhouse lamp in which $F=0.895$ cm.^{2}, $R=0.780$, and $C=0.0000169$.

Series I.	Series II.	Series III.
$\iota =$ 1.4458 amp.	1.5107 amp.	1.5615 amp.
$\Delta P =$ 60.52 v.	62.80 v.	64.74 v.
$W =$ 41.86 ohms	41.57 ohms	41.46 ohms
$E =$ 87.50 w.	94.87 w.	101.09 w.
$H =$ 28.01 c.p.	35.03 c.p.	42.26 c.p.
$E_1 =$ 3.12 w.	2.70 w.	2.39 w.
$T =$ 1580°	1596°	1610°

The resistances were as follows: —

Hours.	Ohms.	Ohms.	Ohms.
0	41.86	41.57	41.76
3	41.86	41.59	41.47
7	41.87	41.56	41.49
10	41.85	41.58	41.52
20	41.86	41.60	41.57
26	41.85	41.59	41.59
30	41.87	41.59	41.62

In a similar manner a lamp of the Berlin Company was tested, in 1888 at 1590°. It was found that the resistance of this lamp remained constant for thirty hours. At 1603°, on the contrary, it increased decidedly, rising to larger and larger values. It appears, therefore, that the temperatures at which various specimens of carbon begin to vaporize are somewhat different.